Journal of Statistical Physics, Vol. 121, Nos. 5/6, December 2005 (© 2005)
DOI: 10.1007/s10955-005-7581-1

The Mathematical Physics of Music

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Received November 22, 2004; accepted June 21, 2005

KEY WORDS: Music; scales; chords; microtonality.

1. PREAMBLE

During the two years that Mitchell Feigenbaum spent with me in the physics department at Virginia Tech, most of our spare time together was devoted to our two mutual passions, drinking fine wines and listening to music. One Saturday, in fact, after listening to the Metropolitan Opera broadcast of *Die Walküre* while drinking a couple of bottles of Schloss Vollrads spätlese, we discovered that another radio station was broadcasting the opera on time-delay, so we listened to it again! (By the way, it's about 5 h long.) I can't remember whether or not we consumed any more spätlese, but if we did Mitchell supplied it since we carried out the second half of this drunken musical orgy at his apartment, my wife by then having thrown us out of our house.

Well, I'm not going to write about the mathematics or physics of wine, even though a lot could be said along those lines. However, mathematics/physics and music are inextricably intertwined,[(1)] and so I'll stick to that subject. Just as examples, Johannes Kepler used musical harmony as a basis for his cosmology. Mersenne, the discoverer of Mersenne primes, elaborated on Kepler's work, but rejected celestial harmony as a model for the solar system.[(2)] This paper is written for those with little or no knowledge of music; others (such as Mitchell) might want to skip directly to Sec. IV.

2. SCALES AND ASSOCIATED INTERVALS

Traditionally in western music, the octave, which ranges from frequencies f to $2f$, is divided into twelve intervals called "semitones." These

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correspond to the intervals between the 12 keys (seven white and five black) in one octave of a piano keyboard. The explanation for this division of the octave is hidden in the mists of Indo-European prehistory. (But I suggest mathematical reasons for this choice in Sec. 4). Other civilizations use different divisions; for example Near Eastern music frequently employs quarter tones while the Slendro and Pelog scales of Javanese music are unique to that culture.[3] In any event, I shall consider only western music here.

When twelve notes separated by semitones are sounded successively, the result is the "chromatic scale." In general, the term "scale" refers to an ordering of notes from the lowest pitch to the highest.[4] The notes of the chromatic scale are designated by the letters A through G with or without superscripts $^{\#}$ (sharp) and $^{\mathrm{b}}$ (flat). The interval between the note X and $X^{\#}$ or X^{b} is one semitone, with the sharped note higher and the flatted note lower. Although enharmonic notes like $\mathrm{C}^{\#}$ and D^{b} are musically different, in the (equal-tempered) tuning scheme adopted in this paper they are the same note, represented on a piano by the black key lying between the white keys C and D. In this tuning scheme each semitone is one twelfth of an octave that is, using note symbols to represent frequencies, $X^{\#}/X = X/X^{\mathrm{b}} = 2^{1/12}$. It is convenient to use a logarithmic unit of frequency called the cent: two notes of frequency f_1 and $f_2 > f_1$ are separated by a number of "cents" given by the formula

$$\text{cents} = 1200 \log_2 f_2/f_1 = 1731\ \ln f_2/f_1$$

So there are 1200 cents in an octave; each (tempered) semitone contains 100 cents. Using the symbols S to represent a semitone and T a whole tone (two semitones) the so-called "major scale" can be represented by the symbolic sequence

TTSTTTS.

Note that this comprises 1200 cents, that is one octave. We can say that the scale begins and ends on the same note if we identify the notes X and the note X' one octave higher, thus speaking of the equivalence class X. This is a true equivalence relation, as can be easily verified; musicians have long known this, and refer to the "pitch class" X.

The sequence given above for the major scale can be realized on a piano by playing eight successive white keys beginning with C. There are clearly seven similar scales obtained by starting with a different white key. These are the celebrated "ecclesiastical modes" of medieval music, each sequence being obtained from the major-scale sequence above by cyclic permutation. Today only the major and minor modes are of primary importance; the symbolic sequence for the (natural) minor scale is

TSTTSTT.

All these seven-note modes are called *diatonic* scales. They originated from the Greek reduced scales of four notes covering half an octave, the so-called "tetrachords." Much later, in Christian ecclesiastical music (4th century) two tetrachords became superimposed to form a *diatonic scale* of seven notes (in interval arithmetic: $4+4=7$).[(5)] At the same time the concept of the *tonic* appeared: this is the base note of the scale, to which a melodic line must return on order to release the tension that builds as the melody strays from it. (Musicians call this return to the tonic "resolution.") The seven notes of the superimposed tetrachords are called, technically, a "diatonic subset" of the 12-note chromatic scale. Not every 7-note subset is diatonic; later I shall discuss the conditions required for a set to be diatonic. There are also five-note "pentatonic" sets, and corresponding scales, the pentatonic scale being thought to be more primitive than the diatonic.[(6)]

In addition to chromatic, diatonic and pentatonic scales, there is a "whole tone scale" consisting of six consecutive whole tones. Whole tone scales were used extensively by the French impressionists, especially by Claude Debussy, whose opera *Pelléas et Mélisande* is permeated by whole-tone music.

The scales give rise to the names of the intervals commonly used in musical notation. The interval C:G is called a (perfect) fifth because G is the fifth note in the major (or minor) scale beginning on the note C. It contains seven semitones. The interval C:F containing five semitones is called a perfect fourth for a similar reason. The interval C:E (four semitones) is a Major third because E is the third note in the C Major scale. An interval one semitone smaller than a major interval is called a "minor interval;" for example C:E^b is a minor third (note that E^b is the third note of the c minor scale). An interval smaller than a minor interval by another semitone is known as a "diminished interval." Thus C:E^{bb} would be a diminished third. Although E^{bb} and D are the same notes on the piano, they are in fact different notes musically. It would be incorrect to call C:D a diminished third. It is in fact a major second.

An interval one semitone larger than a Major interval is called an "augmented interval;" thus C:$E^{\#}$ (note: *not* C:F) is an augmented third.

Perfect intervals can be diminished (reduced by a semitone) or augmented (increased by a semitone) only. The diminished fifth, C:G^b and the augmented fourth C:$F^{\#}$ are the two tritones (containing six semitones, i.e.

three whole tones, and thus occuring in the whole-tone scale. The tritone is so dissonant that Medieval monks called it *diabolus in musica*.

3. CHORDS

A chord is a combination of three (or more) notes, sounded simultaneously (or successively as an "arpeggio.") Normally, one begins with a scale, let's say for the sake of illustration the C Major scale C-D-E-F-G-A-B-(C′), and forms chords by combining the notes n, n+2 and n+4. (As necessary, continue into the octave beginning with C′.) Thus corresponding to this C Major scale are seven basic chords, C-E-G, D-F-A, etc. The chords consisting of a Major third followed by a minor third (C-E-G, F-A-C and G-B-D) are called "Major chords." A "minor chord" consists of a minor third followed by a major third: D-F-A, E-G-B and A-C-E. The remaining chord, B-D-F consists of two minor thirds superimposed. It is called a "diminished chord." Let me also mention "augmented chords" consisting or two major thirds and the "diminished seventh chord," three minor thirds. While of lesser importance in music than the chords already mentioned, these chords are important in the mathematical analysis of scales.

Of special importance are the chords constructed on the first degree of the scale called "one chords" (also called "tonic" chords). The four chord is called the "subdominant" and the five chord the "dominant".(7) In a Major scale these are all Major chords; that's why the sequence TTSTTTS is called a Major scale. In minor key (TSTTSTT) the three principal chords are all minor chords. And that's why *that* sequence is called a minor scale.

Note that all seven of the scales (or modes) corresponding to cyclic permutations of the basic symbolic sequence TTSTTTS contain three major chords, three minor chords and a diminished chord. However, only the Major (resp. minor) scale has Major (resp. minor) chords as tonic. subdominant and dominant. This is one important reason that the other five modes have all but vanished from today's music.

Chord "inversions" are simply the three notes arranged in a different order. For example, the Major chord C:E:G has, as its first inversion E:G:C′ and as its second G:C′:E′. The first inversion is sometime called a 6:3 chord and the second inversion a 6:4 chord, from the intervals involved. (By the same token the original chord—the "root position"—may be called a 5:3 chord.) Whatever they are called, inversion does not change the root of the chord. In fact, the properties of all the inversions of a chord are pretty much identical, a fact discovered by Francois Couperin (1668–1733).

4. THE "CIRCLE OF FIFTHS" AND THE GROUP $C_{12}^{(8)}$

Let us denote the notes of the scale by numbers rather than letters, $C = 0$, $C^{\#} = D^{b} = 2$, and so forth up to $B = 11$. Recalling that the note X really represents the equivalence class of all X's, then C and C′, for example, are both represented by the number 0. Then the 12 numbers 0–11 become elements of the cyclic group C_{12} where the group operation is addition, mod 12. Evidently, for $n \in C_{12}$ we have $n^{-1} = 12 - n$. This group has two generators (in addition to the trivial generators 1 and $1^{-1} = 11$) namely 7 and its inverse 5. (g is a generator of a group G of n elements if $G = \{I, g, g^2, g^3, \ldots, g^{n-1}\}$ where I is the identity element of G. In our case, of course, $I = 0$.)

The homomorphism of C_{12} generated by 7 is known as the "circle of fifths." It is an important musical tool, known to every beginning music student:[9]

$$0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, (0)$$

The homeomorphism generated by 5, which is just the above in reverse order, might be called the circle of fourths (but it usually is not; cf. below).

Note that the numbers in the circle of fifths represent the number of semitones that the corresponding note lies above C (so C:G, the perfect fifth, is seven semitones as we already know from Sec. II). The direction from left to right in (8) is called the "dominant" or sometimes "sharps" direction (because in the key of $C = 0$, $G = 7$ is the dominant). Similarly the backward direction 0, 5, 3, etc. is called the "subdominant" or sometimes "flats" direction because $F = 5$ is the subdominant in the key of C. Melodic motion around the circle of fifths in the dominant direction is supposed to increase the mental tension seeking a return to the tonic while motion back in the subdominant direction is supposed to relieve this tension.

However our goal here is to try to understand the mathematical structure of music by studying C_{12}. This group has subgroups C_2, C_3, C_4, and C_6. These subgroups (or better, their quotient groups with respect to C_{12}) all have musical meanings. Thus C_{12}/C_2 are the six tritones 0–6 (i.e. C:F$^{\#}$), 1–7, 2–8, etc. C_{12}/C_3 the four diminished-seventh chords 0–3–6–9, etc. C_{12}/C_4 are the three augmented chords 0–4–8, etc. while C_{12}/C_6 are the two whole-tone scales, 0–2–4–6–8–10, etc.

Any seven connected (i.e. successive) elements in the circle of fifths is a *diatonic subset* of C_{12}.[8,10] Such a set enjoys what Balzano called the "$F \rightarrow F^{\#}$ property," i.e. moving from one such set to the next adjacent one leaves six notes the same and changes the seventh by a minimal amount, up or down a semitone depending upon the direction of motion. So for example, the connected set

$$5, 0, 7, 2, 9, 4, 11$$

represents the C-Major scale or one of its modes depending on the (cyclic) order chosen. (Start with the tonic note and ascend in numerical order.)

The pentatonic scale consists of five connected elements in the circle of fifths, and it also has the $F \rightarrow F^{\#}$ property, so five connected elements might be called a pentatonic set. *No other connected set of the circle of fifths than the diatonic or penatonic has this $F \rightarrow F^{\#}$ property*. This important property for a musical scale is at the root of the Western use of diatonic (and pentatonic) scales.

The group C_{12} is isomorphic to $C_3 \times C_4$. Actually, this is a special case of a more general result:

Proposition 1. For every positive integer j, the group $C_{j(j+1)}$ is isomorphic to $C_j \times C_{(j+1)}$.

Proof. Clearly $C_{j(j+1)}$ and $C_j \times C_{(j+1)}$ have the same number of elements. To prove isomorphism it is sufficient to prove that $C_j \times C_{(j+1)}$ has an element of period $j(j+1)$.[11] Clearly 1 is of period j in C_j and of period $j+1$ in $C_{(j+1)}$. Therefore (1,1) is an element of $C_j \times C_{(j+1)}$ with period equal to the lowest common multiple of j and $j+1 = j(j+1)$. ∎

The isomorphism means, for the case $j = 3$, that every integer (mod 12) can be written in the form $3n + 4\mathrm{m}$, $0 \leqslant n \leqslant 3$ and $0 \leqslant \mathrm{m} \leqslant 4$. (For example $5 = 3 \times 3 + 4 \times 2$, mod 12.) Musically, it means that every note of the chromatic scale can be written as a combination of Major and minor thirds (mod 12).

According to Schonberg,[12] it was Ferrucio Busoni in 1911 who suggested the possibility of dividing the octave into more than 12 intervals, so-called "micro-tonality." Since Busonis's suggestion there have been numerous attempts to create a microtonal scale by such well-known composers as Easley Blackwood and Ben Johnston. The advent of electronic music has made micro-tuning more feasible (although earlier composers like Busoni and Johnston were thinking in terms of ordinary instruments, like strings). Most early attempts at micro-tuning, in particular Blackwood's, were visualized as means of optimizing chordal consonance. But Balzano[8] took the tack of looking for N-tone scales with a group-theoretical structure similar to that of the 12-tone scale. He was able to prove that only for $N = j(j+1)$ for some positive integer j is the $F \rightarrow F^{\#}$ property possible. This along with Proposition 1, led him to study

N-tone scales for $j = 4$ and 5. He proved the following proposition (the proof, in somewhat simpler form, is also found in ref. 10).

Proposition 2. $C_{j(j+1)}$ has generators $2j + 1$ and its inverse $j^2 - j - 1$.

The connected subsets of length $2j + 1$ and $j^2 - j - 1$ in the automorphism generated by $2j + 1$ (or $j^2 - j - 1$) are the sets which enjoy the $F \rightarrow F^{\#}$ property and are the analogues of the diatonic and pentatonic scales respectively. Many properties of these microtonal scales are discussed in the references by Balzano and Zweifel previously cited to which the curious are referred. For example, by identifying the analogue of Major and minor thirds, it is possible to use Proposition 1 to identify the analogue of the Major and minor modes in these microtonal scales. Etc.

At my instigation, the composer Jon Polifrone has written some microtonal music which was performed on a computer by Maureen Mellody as part of her senior honor's thesis at Virginia Tech. Tapes of this music are available: contact the author: zweifel@vt.ed. (A referee asked that I discuss what this music "sounds like," a task I am incapable of performing. Suffice it to say that it sounds different from any other music I have ever heard; the generalized intervals, like the "thirds" and the "fifths" have no apparent consonances.)

A much longer version of this article, covering additional topics like origin of consonance, tuning methods, etc. is also available from the author.

REFERENCES

1. In the Middle Ages music was part of the Quadrivium, the higher division of the seven liberal arts. Music was thus grouped with arithmetic, astronomy and geometry and hence, evidently, considered a science. (The lower division, the trivium, consisted of grammar, rhetoric and logic.)
2. Thus "The music of the spheres." John Fauvel, Raymond Flood and Robin Wilson, eds. *Music and Mathematics*. Oxford University Presss, 2003. Chapter 2.
3. Donald E. Hall, *Musical Acoustics*, 2nd Ed. (Brook/Cole Publishing Company, Pacific Grove, CA 1980), p. 407.
4. An abuse of nomenclature has crept in, namely referring to the frequency (or "pitch") of a note. In first approximation, this means the frequency of the fundamental. However, a few instruments, e.g. tympani and some gongs, produce no fundamental; the ear constructs the pitch from the overtone series. (See Ref. 3, pp. 429–430.) It is the frequency of the fundamental, real or implied, that I mean when I write of the frequency of a note.
5. See Hermann Helmholtz, *On the Sensations of Tone* (2nd English Ed. Dover, New York, 1954). pp. 237 ff. It is possible, although unproved, that the Greeks had already used diatonic scales (at least whoever invented the Greek names like "Ionian," "Dorian," etc. for the diatonic modes must have thought so).

6. A pentatonic scale can be played on the five black keys of a piano (just as a diatonic scale can be played on the seven white keys). A familiar example of a pentatonic melody is *Auld Lang Syne*. Richard Wagner used pentatonic melodic lines in *Der Ring des Nibelugen* when he was dealing with voices of nature, for example the Forest Bird in Act II of *Siegfried*.
7. The fifth degree of every mode is called the "dominant" because in plainsong chant it was the usual reciting tone.
8. Much of the treatment in this section was taken from Gerald J. Balzano, *Computer Music J*. **4**:66–84 (1980).
9. *Inter alia*, it gives the order in which sharps and flats appear in key signatures. (The "key" of a scale is simply its tonic note.)
10. Paul F. Zweifel, *Perspectives of New Music* **34**:140–161 (1996).
11. Frank J. Budden, *The Fascination of* Groups, Cambridge University Press (1972). Chapter 23. I am indebted to a referee for pointing out that this proposition is special case of the known result that if q and p are relatively prime then the groups $C_p \times C_q$ and C_{pq} are isomorphic. She or he pointed out further that the equivalent result in number theory is sometimes called the Chinese Remainder Theorem.
12. Harold C. Schonberg, *The lives of the great composers*. W. W. Norton and Co., New York (revised edition, 1981). p. 554.